Self-organized relaxation in a collisionless gravitating system

Yasuhide Sota,[1,2,*] Osamu Iguchi,[1,†] Tohru Tashiro,[1,‡] and Masahiro Morikawa[1,§]
[1]*Department of Physics, Ochanomizu University, 2-1-1 Ohtuka, Bunkyo, Tokyo 112-8610, Japan*
[2]*Advanced Research Institute for Science and Engineering, Waseda University, Ohkubo, Shinjuku-ku, Tokyo 169-8555, Japan*
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We propose the self-organized relaxation process which drives a collisionless self-gravitating system to the equilibrium state satisfying local virial (LV) relation. During the violent relaxation process, particles can move widely within the time interval as short as a few free-fall times, because of the effective potential oscillations. Since such particle movement causes further potential oscillations, it is expected that the system approaches the critical state where such particle activities, which we call gravitational fugacity, is independent of the local position as much as possible. Here we demonstrate that gravitational fugacity can be described as the functional of the LV ratio, which means that the LV ratio is a key ingredient estimating the particle activities against gravitational potential. We also demonstrate that the LV relation is attained if the LV ratio exceeds the critical value $b=1$ everywhere in the bound region during the violent relaxation process. The local region which does not meet this criterion can be trapped into the presaturated state. However, small phase-space perturbation can bring the inactive part into the LV critical state.

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I. INTRODUCTION

It is well known that collisionless relaxation process plays a key role in driving the gravitational objects to the equilibrium state observed as elliptical galaxies or dark halos in our universe. Such relaxation process has been well analyzed with *N*-body simulations from the context of the formation of density profile following $r^{1/4}$ law of elliptical galaxies [1,2] or dark halo formations after in-falling from the cosmological background [3–5].

As a relaxation process toward such an equilibrium state, phase mixing and violent relaxation has been proposed as a collisionless relaxation process, which, if completed, leads a self-gravitating system (SGS) to the entropy maximum state called Lynden-Bell distribution [6]. This distribution, however, cannot directly be applied to three-dimensional open models, since it has an infinite mass and energy against the state attained through *N*-body simulations. Actually, in numerical simulations, the violent relaxation is not completed and the state reaches the equilibrium state which cannot be described by the Lynden-Bell statistics, mainly due to the existence of the particles with positive energy which escape to infinity [7,8].

Then how can we characterize such a quasiequilibrium state in an open system where particles can evaporate infinitely? Recently, we have numerically shown that the bound states after a cold collapse or cluster-cluster collisions are virialized not only globally but also locally for a wide range of initial conditions [9–11]. Such a state with the local virial (LV) relation is not a general solution for the stationary state of the Vlasov equation. For example, Plummer's model is a unique solution satisfying the LV relation among the class of polytropes. In addition, it is special among the class, since it has the unique solution with infinite extension of particles and with finite total mass [12]. From such viewpoints, Plummer's model was originally analyzed by Eddington [13] and was generalized to several families of both analytical and numerical solutions with anisotropic velocity dispersion [11,14].

Provoked by the remarkable characters such as the LV relation, Eddington tried to show that Plummer's model is the local minimum of the *H* function, but this approach was not fully successful [13]. So what kind of principle can characterize the LV relation as an attractor? In general, not all of the stationary state can be explained by the maximum entropy principle. For example, the nonequilibrium state is attained in the system sustained in the energy flow. In such open systems, energy injection is balanced with the local energy dissipation.

During the collisionless stage, potential oscillations play a key role in violent relaxation. Since such oscillations are induced from the particle movements in the bound region, the activity of the particle against gravitational potentials is a key ingredient for the relaxation process. In this paper we propose the gravitational fugacity which quantifies such particle activities against gravitational potential. We will also show that the fugacity can be described as the functional of the LV ratio, which means that the LV ratio is the indicator estimating the local activity of particles against gravitational potential. This means that the collisionless relaxation process of SGS is induced not from the entropy maximum principle minimizing the local temperature fluctuation, but rather from that minimizing local fluctuation of particle fugacity.

In Sec. II, we define the local fugacity of particles as the generalization of local evaporation rate. In addition, we will show that the fugacity can be the functional of the LV ratio, which directly connect the LV ratio to particle activities against gravitational potential. We will also generalize it to the case with the anisotropic Gaussian velocity distribution and show that the local fugacity is sensitive to the LV ratio but is not so relevant with the anisotropy. In Sec. III, we will numerically investigate the collisionless relaxation process

*sota@cosmos.phys.ocha.ac.jp
†osamu@phys.ocha.ac.jp
‡tashiro@cosmos.phys.ocha.ac.jp
§hiro@phys.ocha.ac.jp

for several initially spherical models and see that the LV relation is attained for the local region which experiences the high fugacity state where the fugacity is beyond the critical value. In Sec. IV, we will see characters of the LV relation through several tests to find the criterion to achieve the LV critical state. Finally, in Sec. V, we will summarize our analysis and comment on the analogy of the LV relation with the critical state of self-organized criticality (SOC) or the role of the LV relation from the viewpoints of superstatistics.

II. LV RELATION AS THE STATE WITH CONSTANT FUGACITY UNDER GLOBAL VIRIAL CONDITION

In general, gravitational relaxation is mainly divided into two processes. First is the collisionless relaxation process which is induced not through the particle-particle interaction but through the oscillation of gravitational potential. Once the system is globally virialized, the relaxation process is ceased until two-body interactions are effective and the system begins to evolve toward two-body relaxation. The particle evaporation rate defined as the rate of particles whose speed exceeds the local evaporation rate is a proper indicator for such collisional relaxation process, since the evaporation of particles occurs only through the two-body interaction during the stationary collisionless stage.

During the stage of violent relaxation, on the other hand, evaporation rate is not necessarily a key ingredient for the relaxation process, since the two-body interaction is negligible during the period. In fact, for as long as we examined, the particles with, positive energy emerged only at the moment of the maximum collapse with very low initial virial ratio. However, during the violent relaxation, particles can be activated through the potential oscillations. The number of particles which gain energy enough to move far away from the potential minimum increase even if they cannot escape infinitely from the gravitational center.

In this section first we define the gravitational fugacity as the quantity which represents such particle activities against gravitational potential. Then we will show that the fugacity can be described as the functional of the LV ratio under the assumption that the local velocity dispersion is isotropic. Finally we will also investigate how strongly the velocity anisotropy affects the relation between the gravitational fugacity and the LV ratio. We will show that the anisotropy affects the gravitational fugacity less sensitively than the LV ratio. This means that the LV ratio becomes an effective indicator to estimate the particle activities against gravitational potential.

A. Relation between the LV ratio *b* and the gravitational fugacity for isotropic Gaussian velocity distribution

In previous papers [9–11], we showed that the LV relation is well realized for cold collapse simulations starting from a homogeneous sphere. In these simulations, the velocity distribution turns out to be locally Gaussian with different temperature on each shell after a cold collapse with strong violent relaxation [16]. Hence, after a cold collapse, the velocity distribution is locally well approximated by Gaussian with the velocity dispersion $\sigma^2(\vec{r})$ depending on the local position $\vec{r}$. In this case, the speed v of each particle at the position $\vec{r}$ is governed by the following phase-space density:

$$f(v,\vec{r}) = \rho(\vec{r})\left(\frac{3}{2\pi\sigma^2(\vec{r})}\right)^{3/2} 4\pi v^2 \exp\left(-\frac{3v^2}{2\sigma^2(\vec{r})}\right), \quad (1)$$

where $\rho(\vec{r})$ is a mass density at $\vec{r}$. With the velocity dispersion $\sigma^2(\vec{r})$ and the local potential energy $\Phi(\vec{r})$, the LV ratio is defined as follows [11]:

$$b(\vec{r}) := -2\sigma^2(\vec{r})/\Phi(\vec{r}). \quad (2)$$

Since the total energy of a particle evaporating from this system must be positive, we can assess the lowest speed $v_{\rm cr}$ of the evaporating particles,

$$v_{\rm cr} = \sqrt{-2\Phi(\vec{r})}. \quad (3)$$

In the local volume dV at $\vec{r}$, the total mass of the particles whose speeds exceed $av_{\rm cr}$ with the constant value a can be described as

$$M_a(\vec{r}) = dV\rho(\vec{r})R_a(\vec{r}), \quad (4)$$

where R_a is the rate of the particles whose speeds exceed $av_{\rm cr}$, which is described as

$$R_a(\vec{r}) = \frac{1}{\rho}\int_{av_{\rm cr}}^{\infty} f(v,\vec{r})dv = 1 - \mathrm{Erf}\left[a\sqrt{-\frac{3\Phi}{\sigma^2}}\right] + 2a\sqrt{-\frac{3\Phi}{\pi\sigma^2}}e^{3a^2\Phi/\sigma^2}, \quad (5)$$

where Erf[$\cdots$] is the error function.

By setting Eq. (2) into the right-hand side of the above equation, it becomes obvious that R_a is a functional of the LV ratio b,

$$R_a[b(\vec{r})] = 1 - \mathrm{Erf}\left[a\sqrt{\frac{6}{b(\vec{r})}}\right] + 2a\sqrt{\frac{6}{\pi b(\vec{r})}}e^{-6a^2/b(\vec{r})}. \quad (6)$$

Note that R_1 is identified with the local evaporation rate $R_{\rm ev}$. In this case, R_a indicates the rate of the particles which spread infinitely. When $a<1$, on the other hand, R_a includes not only the particles escaping infinitely but also those trapped in the finite region. For larger value of a, the particle can move far away from the gravitational center, since R_a includes the particles with the higher value of kinetic energy against the absolute value of the local potential. Therefore, the value of a offers the lower bound of the typical scale L_a to which the particle at $\vec{r}$ can move away maximally in the future. As the local temperature increases against gravitational potential, the particle activities are enhanced and the value of R_a increases for all of the values of a. Hence, we call R_a gravitational fugacity, since it represents the particle activities against gravitational potential. From the fact that R_a is the functional of b in Eq. (6), the LV relation $b(\vec{r})=1$ is induced from the global virial condition and the condition that R_a is constant everywhere (see the Appendix). This means that the SGSs self-organize themselves so as to make the particle activities constant everywhere. In the next sec-

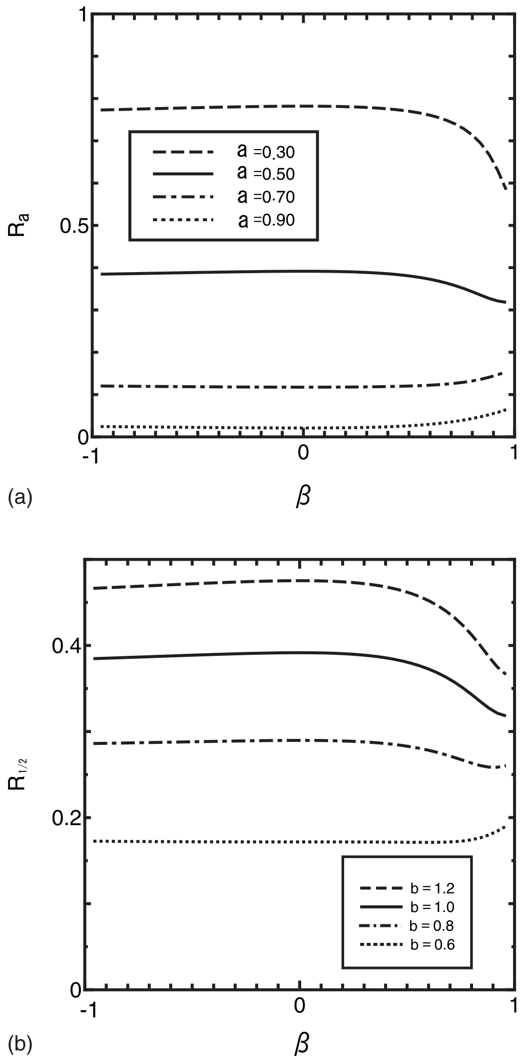

FIG. 1. (a) R_a for the LV ratio $b=1$ as a function of β. Each line represents the case with $a=0.3$ (dashed), 0.5 (solid), 0.7 (dotted-dashed), 0.9 (dotted). For all of the values of a, R_a is almost constant except for the high value of β. (b) R_a for $a=1/2$ as a function of β. Each line represents the case with $b=0.6$ (dotted), 0.8 (dotted-dashed), 1.0 (solid), 1.2 (dashed). For all of the values of b, $R_{1/2}$ is almost constant except for the high value of β.

tion, we will numerically investigate R_a for $a=1/2$, since it is the typical value measuring the particle activities against gravitational potential. We will show that the LV relation is attained for the spherical shell where the gravitational fugacity exceeds the critical value $R_{1/2}[b=1]$.

B. Relation between the LV ratio b and the gravitational fugacity for anisotropic Gaussian velocity distribution

In the above derivation of the LV relation, we assume that the velocity distribution is locally isotropic everywhere during the violent relaxation process. However, it is well known that velocity dispersion becomes anisotropic just after a cold collapse [1]. Hence it seems important to examine how strongly the gravitational fugacity depends on the anisotropy. Here in order to ascertain this, we obtain the gravitational fugacity for anisotropic Gaussian velocity distribution, where the velocity dispersion in tangential direction is different from that in radial direction. In the cylindrical coordinates for velocity space, the phase-space density can be described as

$$f(v_r,v_t,\vec{r}) = \sqrt{\frac{2}{\pi}}\frac{v_t\rho(\vec{r})}{\sigma_t^2(\vec{r})\sigma_r(\vec{r})}\exp\left(-\frac{v_t^2}{\sigma_t^2(\vec{r})}-\frac{v_r^2}{2\sigma_r^2(\vec{r})}\right), \tag{7}$$

where v_r and v_t are radial and tangential velocity components and σ_r^2 and $\sigma_t^2:=\sigma_\theta^2+\sigma_\phi^2$ are velocity dispersion of radial and tangential components, respectively. The gravitational fugacity for this function can be evaluated as

$$R_a(\vec{r}) = \frac{1}{\rho}\int\int_{v_r^2+v_t^2\geq a^2v_{\rm cr}^2} f(v_r,v_t,\vec{r})dv_r dv_t. \tag{8}$$

Substituting (7) into (8) and using the anisotropy parameter defined as

$$\beta:=1-\frac{\sigma_t^2}{2\sigma_r^2}, \tag{9}$$

the fugacity can be described as the functional of both b and β as

$$R_a[b,\beta] = 1-\frac{2}{(1-\beta)\sqrt{\pi}}\left(\frac{3-2\beta}{3}\right)^{3/2}\int_0^{a\sqrt{6/b}}x^2 \times\exp\left(-\frac{3-2\beta}{3(1-\beta)}x^2\right)G\left[\frac{\beta(3-2\beta)}{3(1-\beta)}x^2\right]dx, \tag{10}$$

where $G[y]:=\int_{-1}^{1}\exp(yt^2)dt$, which correctly reduces to (6) when $\beta=0$.

As is shown in Fig. 1, R_a, both for the fixed value $b=1$ and for the fixed value of a, is almost constant as a function of β except for the larger value of β. In our numerical results, the velocity dispersion is not highly radially anisotropic at least within a one-half-mass radius [9,11]. Hence we can roughly say that the gravitational fugacity is mainly determined by the b value in the bound region, and the fluctuation is minimized in the LV relation even if we take account of the anisotropy of velocity dispersion.

III. SELF-ORGANIZED RELAXATION PROCESS INDUCED FROM GRAVITATIONAL FUGACITY FOR N-BODY COLD COLLAPSE

As is shown in our previous papers [9–11], the LV relation is attained for several classes of cold collapse simulations. Here in this section, we will first overview the results of N-body simulations with spherical initial conditions. We will see that the LV ratio oscillates around the critical value $b=1$, then converges to it for the collapse of the initial homogeneous sphere. For the initial cuspy density profile, on the other hand, the LV ratio in the central part retains the lower value.

We will also see that such differences among initial conditions are due to the behavior of gravitational fugacity. Here we estimate the value of fugacity $R_{1/2}$ by counting the number of particles whose velocity exceeds the value $0.5v_{\rm cr}$ on each shell, without any assumption for the function form of velocity distribution. For cold collapse simulations from a homogeneous sphere, we will see the fugacity pass over the critical value everywhere at the moment of maximum collapse. For the collapse of cuspy density profile, on the other hand, the central part of the bound region is not so activated that the fugacity keeps lower than the critical value there. The LV ratio behaves in the same way as the gravitational fugacity for both of the simulations.

A. Numerical setting and realization of the LV relation for several cold collapse simulations

Here we overview the numerical setting of our simulations and our numerical results for the achievement of the LV relation. For N-body simulations we use the unit of $G=M=r_s=1$, where M and r_s are the total mass and the radius of initial sphere, respectively, and the initial free-fall time $t_{ff}:=\sqrt{r_s^3/GM}$ as the time unit for the time-sequence of physical variables. The simulations are performed on GRAPE-5 or GRAPE-7 for all of our runs [17], in which the potential and force of interaction are softened through the softening length ϵ using the Plummer softening. As for the initial conditions, we generate the spatial coordinates of particles randomly within a sphere so as to follow the initial mass density. The coordinates in velocity space are generated from the Gaussian distribution whose variance is adjusted according to the value of $V_{\rm in}$. Time step of numerical integration is fixed to the value for which the total energy is conserved to better than 10^{-3}.

For the evaluation of the LV ratio, we divide the bound region composed of the particles with negative energy into plural concentric shells and measure the averaged value on each shell between $t=5t_{ff}$ and $10t_{ff}$. Fluctuations of LV ratio during the period are depicted as the error bar in the following figures, which represents the rms of LV ratios between the time intervals. The reliability of our numerical results is examined by changing the softening length ϵ. We found that the LV ratio b takes the critical value $b=1$ quite well for smaller value of $\epsilon \lesssim 2^{-6}$, although it deviates downwards for $\epsilon > 2^{-6}$ (Fig. 2). These results suggest that the LV relation is well attained through the gravitational interaction as long as the softening length is sufficiently small. Hereafter we will fix the softening ϵ to 2^{-8} and check the dependence of the achievement of the LV relation on the particle number N and on the initial virial ratio $V_{\rm in}$ [Figs. 3(a)–3(c)].

We can see that the LV ratio converges to the critical value $b=1$ quite well in the central part of bound region up to $t=10t_{ff}$, while it oscillates around it in the outer part, especially for the warmer collapse with higher initial virial ratio [Fig. 3(a)]. When we increase the particle number for cold collapse simulations with lower initial virial ratio, we can see that the LV ratio deviates from the critical one up to $t=10t_{ff}$, although it is not fluctuated [Fig. 3(b)]. We can speculate that this deviation comes from the radial instabilities for cold collapse characterized as the spiral arms in phase space [15]. As the authors showed in the paper, such a spiral structure is not stable but rather dissipative. In fact, we can numerically ascertain that the LV ratio approaches the critical value as the time elapses [Fig. 3(d)].

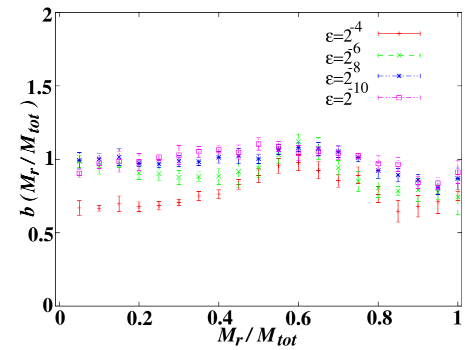

FIG. 2. (Color online) The LV ratio averaged between $t=5t_{ff}$ and $t=10t_{ff}$ derived from a homogeneous sphere with $N=5000$ and initial global virial ratio $V_{\rm in}=0.0$ for several values of softening length ϵ. Each line represents the LV ratio for $\epsilon=2^{-4},2^{-6},2^{-8}$, and 2^{-10}, respectively. The LV ratio for $\epsilon=2^{-4}$ deviates from a critical value $b=1$ around the central part of the bound region.

For the simulations of power-law density profiles, on the other hand, LV ratio keeps the lower value for a long period especially in the central part of the bound region for steeper density profile [Fig. 3(c)]. In fact, we can see that the LV ratio does not pass over the critical value for the shells inside $0.1\ M_{\rm tot}$ even at the moment of the maximum collapse [Fig. 4(b)]. This is a remarkable difference from the case with initial homogeneous sphere, where LV ratio passes over the critical value everywhere at the moment of the maximum collapse and the LV relation is attained [Fig. 4(a)].

B. Time evolution of local fugacity for N-body simulations starting from a homogeneous sphere

For the case starting from a homogeneous sphere with vanishing virial ratio $V_{\rm in}=0$, $R_{1/2}$ passes over the critical value $R_{1/2}[b=1]$ at the moment of maximum collapse on all of the shells (Fig. 5). Once it passes over, it turns out to be reduced, because the particles are too activated to stay there and turn to spread out against gravitational potential. This seems plausible, because just after the moment of maximum collapse, the particles efficiently begin to spread out from the inner region to the outer region by climbing up the potential, because of their high kinetic energies. This state with excessive kinetic energy can be quantified as the high value of gravitational fugacity. Since they lose the kinetic energy as they go up the potential hills, the local fugacity turn out to be reduced. If the averaged fugacity becomes lower than the averaged value, they begin to increase. Hence, the oscillations of fugacity occur until the local fugacity at each position is balanced with each other. Finally it settles down to the distribution consistent with the LV relation $b=1$.

We can see that the convergence to the critical value is very fast for all of the shells in the case with smaller particle number [Fig. 5(a)], while it becomes slower for the outer shell in the case with larger particle number [Fig. 5(c)], which is because of the surviving of a shock created at the

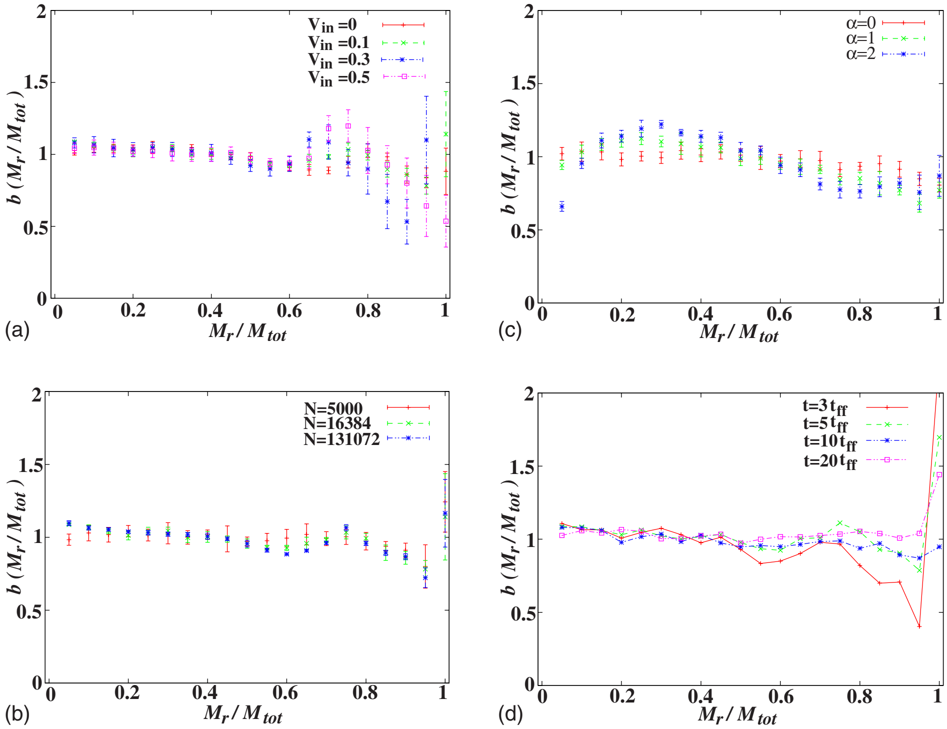

FIG. 3. (Color online) (a) The LV ratio averaged between $t=5t_{ff}$ and $t=10t_{ff}$ derived from a homogeneous sphere with $N=5000$ and initial global virial ratio $V_{\rm in}=0.0,0.1,0.3,0.5$. (b) The same as (a) but with $N=5000,2^{14}$ $(=16\,384),2^{17}(=131\,072)$ and $V_{\rm in}=0.0$. (c) The same as (a) but from spherical density profile $\rho\propto r^{-\alpha}$ with exponent $\alpha=0$, 1, and 2. (d) Snapshots of the LV ratio from a homogeneous sphere with $N=2^{17}$ and $V_{\rm in}=0.0$. Four snapshots at $t=3$, 5, 10, and $20t_{ff}$ are depicted.

maximum collapse. Therefore, such a deviation fades down as the time elapses.

We can also investigate the warmer initial conditions with initial virial ratio $V_{\rm in}=0.1,0.3,0.5$ (Fig. 6). As the initial virial ratio is higher, fugacity on each shell oscillates with a longer time interval. This is because the density of the central core becomes lower for warmer collapse. Since the time scale of the oscillation is determined by the free-fall time determined by the value of the central density, we need to wait longer for a warmer collapse until it settles down to the stationary state.

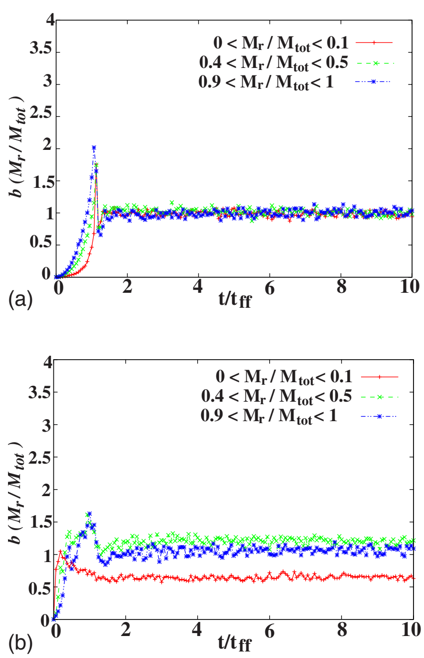

FIG. 4. (Color online) Time sequence of LV ratio b on fixed shell for cold collapse simulations with $N=5000$ and $V_{\rm in}=0.0$ (a) for homogeneous density profile and (b) for cuspy density profile with $\alpha=2$. For both of the figures, the bound region is divided into 10 shells and the b value on the first, fifth, and tenth shells are depicted.

C. Time evolution of local fugacity for *N*-body simulations starting from a cuspy density profile

Next we will investigate the initial cuspy density profile, where the LV ratio becomes less than 1 in the innermost part of the shell, even after the system settles down to the stationary state.

Here we will compare the time evolution of fugacity on several shells. Gravitational fugacity certainly synchronizes with the LV ratio b and keeps lower in the central part (Fig. 7). The fugacity at the innermost shell does not pass over the critical value $R_{1/2}[b=1]$, neither does the LV ratio (Fig. 4). We also checked that the R_a does not pass over the critical value $R_a[b=1]$ for $a<0.5$. This means that the particles in the central region do not spread out to the outer region but stay around the center, which prevents any part of the bound region from realizing the LV relation. This state with excessive potential energy in the central part is highly stable, since it is located in the central core and isolated from the outer part.

IV. CHARACTER OF SELF-ORGANIZED EQUILIBRIUM STATE FROM THE VIEWPOINT OF THE LV CONDITION

In Sec. II, we found that local gravitational fugacity depends on the local position only through the LV ratio b. Hence we can naturally judge the activity of the local region through its LV ratio b. That is, if some parts of the region

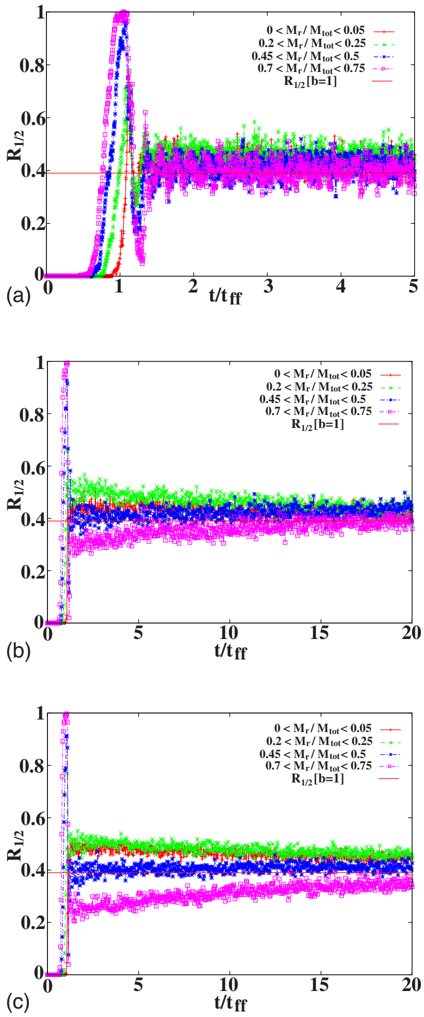

FIG. 5. (Color online) Time evolution of gravitational fugacity $R_{1/2}$ on each shell for the N-body simulations starting from a homogeneous sphere. Each line represents the value on the shell with the mass ratio $M_r/M_{tot}=0.05$ (red), 0.25 (green), 0.5 (blue), 0.75 (pink) with (a) $N=5000$, (b) $N=2^{14}(=16\,384)$, (c) $N=2^{15}(=32\,768)$. The horizontal red line represents the critical value $R_{1/2}[b=1]$.

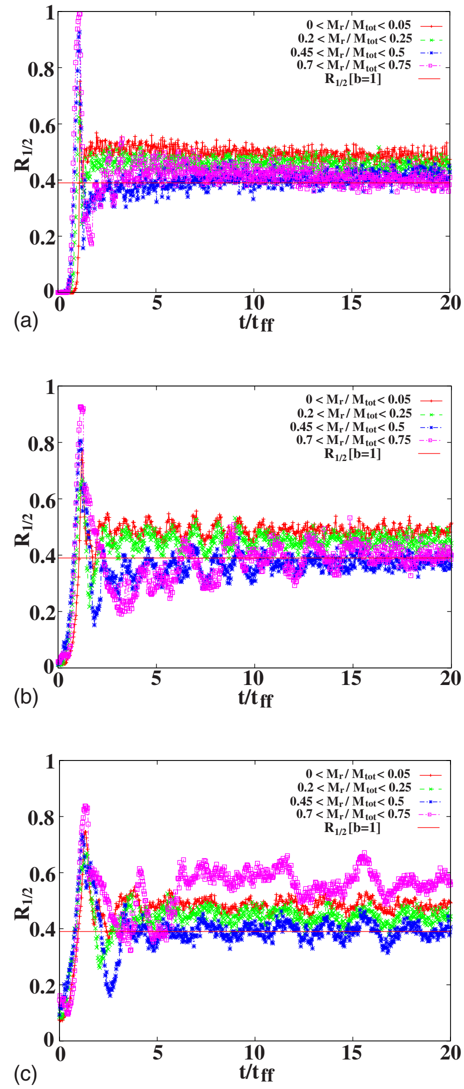

FIG. 6. (Color online) Same as Fig. 5 but for the fixed particle number $N=2^{14}$ and for different initial virial ratios with V_{in} =(a)0.1, (b)0.3, (c)0.5.

take b lower than 1, their activities are less than the averaged value, while if they take b higher than 1, they are more activated.

For the initial cuspy density profile with steeper configuration the central part of the system is not driven to the state with LV relation, but is trapped to the quasiequilibrium state, because it stays in the state with $b<1$. This state may or may not be stable under the perturbation in phase space. In cosmological simulations, this cuspy density profile has common characters with the equilibrium state attained as well as the lower temperature and power-law phase-space density [5,18]. In such cosmological settings, there are a large number of such cuspy halos moving and interacting with each other. Since each of them is influenced by the gravitational forces from other nearby halos, it is worth examining the robustness of such a quasiequilibrium state which does not satisfy the LV relation. Hence, in this section, we examine the stability of both the LV equilibrium state and the quasiequilibrium state especially from the viewpoints of the LV relation.

A. Self-adjusting mechanism through the enhancement of activities for the state with $b<1$

First, we examine a cuspy density profile with the exponent $\alpha=2.0$ for several initial virial ratios (V_{in} $=0.0, 0.5, 1.0$). As is shown in Fig. 8, $b(M_r)$ takes the minimum value at the center of mass and monotonically increases toward outside for these initial distributions. Hence, the particle activities become lower and lower toward the center of mass. Starting from these initial conditions [Fig. 8(a)], we can get the equilibrium state where the central part of $b(M_r)$ is lower than the critical value [Fig. 8(b)]. During this process, the outer part particles tend to spread out because of the high activities while the inner parts tend to collapse because of the lower activities. Hence they do not mix themselves but keep their inner part isolated from the outer part.

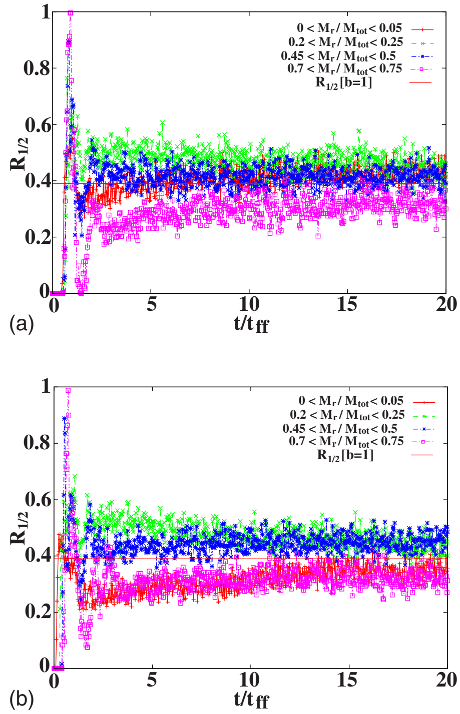

FIG. 7. (Color online) Same as Fig. 5 but for the initial density profile $\rho \propto r^{-\alpha}$ with exponent α= (a) 1 and (b) 2.

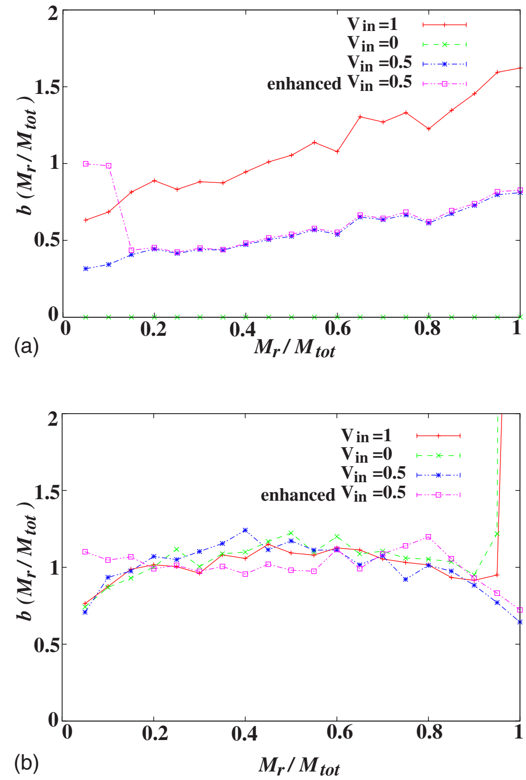

FIG. 8. (Color online) Snapshots of the b distribution as a function of M_r obtained from the cold collapse simulations with N =5000 and $V_{\rm in}$=0.0 for initial cuspy density profile with α=2 at t =0 (a) and $80t_{ff}$ (b). The pink line represents the case that the initial b value in the inner most shell is enhanced to b=1.

On the other hand, if we enhance the LV ratio for the innermost shell up to one, we can enhance the activity of particles against gravitational potential. In this case, not only at the central shell but also at several shells in the inner part, the LV ratio exceeds the critical value at some future moment, which leads to the LV critical state almost everywhere in the bound region [Fig. 8(b)].

From these empirical results, the cuspy density profile turns out to be unstable against the fluctuations which activate the central part of the bound region. From such activations, the system begins to evolve toward the critical state satisfying the LV relation. This means that the state of the cuspy density profile is presaturated and can be evolved to the saturated state satisfying LV relation, against the perturbation which causes strong particle activities in the central part.

B. Robustness of the LV critical state against the LV perturbations

Next, we examine the stability of the LV equilibrium state against the deviations from b=1. Here in order to see this character, we shift the b value of one selected shell for the LV critical state by increasing or decreasing the velocities for the particles in the critical shell at the same rate. This means that the activity of each shell is slightly different from the averaged one. Then we can examine the dissipation of the particles by tracing their positions in later times (Fig. 9).

When we increase the LV ratio b, the fugacity R_a is enhanced and the particles are expanded to the outer shell and the system settles down to the state with b=1 everywhere [Fig. 9(a)]. On the other hand, when we decrease the LV ratio, the fugacity is reduced and the particles fall down to the inner shell and the system again goes back to the state with b=1 everywhere [Fig. 9(b)]. Hence the critical state with b=1 is stable against the shift of the local activities.

V. CONCLUDING REMARKS

In this paper, we propose the gravitational fugacity which represents the activity of particles against gravitational potential. We also show that the gravitational fugacity at each local region is determined directly by the LV ratio at the region. Hence we can pose the physical meaning on the LV relation as the condition that the local activity of particles are balanced with each other on any local region. In fact, we showed that the LV ratio b oscillates around the critical value b=1 and settles down to it when the LV relation is achieved. Especially we found that the LV relation is attained when the LV ratio passes over the critical value b=1 everywhere in the bound region at least once before the system reaches stationary equilibrium state. Hence, the SGS is self-organized toward the critical state with $b(\vec{r})=1$ under the condition that the system passes over the critical state b=1 everywhere. The gravitational fugacity R_a synchronizes with b and converges to the critical value $R_{1/2}[b=1]$, when LV ratio passes over the critical value b=1.

In the cold collapse process starting from a homogeneous sphere, the SGS starts from a state with low fugacity every-

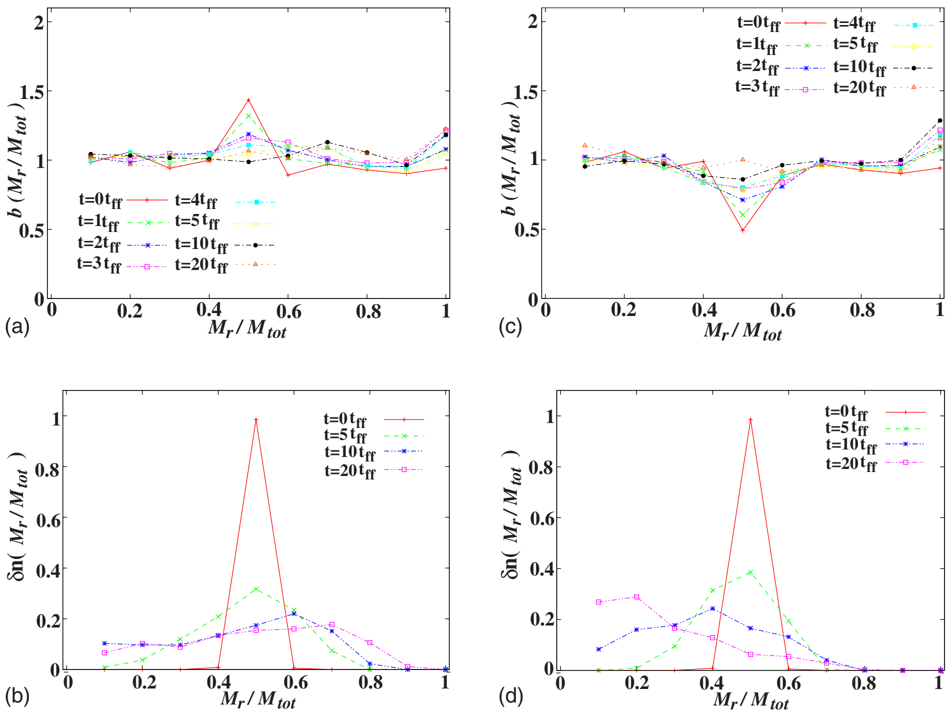

FIG. 9. (Color online) The time evolution of the LV distribution after the LV ratio of one shell is artificially shifted from the critical value after the system reaches the LV state. Here the LV ratio around $M_r=0.5M_{\rm tot}$ is shifted from upper (a) and lower (c) for the equilibrium state starting from a homogeneous sphere with $V_{\rm in}=0.0$ and with $N=5000$. The system smoothly goes back to the critical state $b=1$ within $10t_{ff}$ time interval. The distribution of perturbed particles δn at several moments are also depicted in (b) and in (d) for the upper and lower shift of b, respectively. The particles are smoothly spread toward other shells for both of the cases.

where. Both the LV ratio and the fugacity pass over the critical value largely everywhere at the moment of the initial collapse, which causes violent relaxation. On the other hand, in the cold collapse process starting from a cuspy density profile, LV ratio does not pass over the critical value in the central part of the bound region and most particles stay there until the system reaches the quasiequilibrium state. In this case, the gravitational fugacity R_a with $a\leq 0.5$ does not pass over the critical value $R_a[b=1]$. This is because most particles in the central part fall down into the deep potential well induced from the outside particles. In this case the system stays in the low fugacity state and never reaches the state with the LV relation. Hence we can divide the collisionless relaxation process into two categories: One admitting that the LV ratio passes over the critical value $b=1$ everywhere in the bound region and another which does not. The former self-organizes itself so as to minimize both the LV ratio and the local fluctuation of the gravitational fugacity as much as possible.

This self-organized process with the condition $b>1$ reminds us of the self- organized criticality (SOC) in sandpiles [19]. In sandpiles, avalanches occur whenever the slope exceeds the critical value. In the self-organized process in SGS, particles are well activated before it reaches the state with the LV relation. Such activated particles tend to spread out effectively, which may correspond to the avalanche in sandpiles. Hence it seems worth examining the character of the SGS self-organized process from the viewpoint of SOC, which is characterized by the scaling behavior. Such similarities of the self-organized process with SOC will be well analyzed in our upcoming paper.

In our previous paper [16], we showed that the quasiequilibrium state attained through cold collapse simulations in SGS can be characterized as the velocity distribution superposed with Gaussian distribution with different local velocity dispersion corresponding to the local temperature. The LV relation indicates that the velocity dispersion normalized with the local potential becomes constant and independent of the position. Hence, from the viewpoints of statistical mechanics, we can say that the system settles down not to the isothermal state with constant temperature but to the quasiequilibrium state with constant pseudotemperature normalized with potential. In this critical state, particles can move around the region in different temperatures and reach the equilibrium state. Superstatistics is proposed as such a model superposing the different temperatures although they seem to lack theoretical derivation from the dynamical process [20]. SGS self-organized relaxation may give the hint for explaining the superstatistics from more fundamental dynamical viewpoints.

In cosmological simulations, the stable stationary solution exists for the Jeans equations under the assumption that the ρ/σ^3 follows the scaling law. This solution is special in that it has the property that particles spread infinitely but the total mass of the bound region is finite [5,18]. In fact, this stationary solution can describe the density profile, which was originally proposed by Navarro, Frenk and White (NFW) and is atained in cosmological simulation quite well. . On the other hand, for the cold collapse simulations, the bound state follows the stationary solution of the Jeans equation under the assumption of the LV relation, which also has the property of infinitely spread and finite mass [11,14]. So far the global stability of two solutions has not been compared because they are the solutions under different constraints. Our analysis of gravitational fugacity may give a hint for the stability of these two sorts of solutions. As long as we examined, the cuspy density profile which does not follow the LV relation, is a presaturated state which goes to the LV state if the central isolated region is activated. In actual astrophysical systems, such a transition from the presaturated state to the LV stable state may happen through the sequence of merging processes. In fact, we found that the system approaches the LV state for the merging process, although the time scale of relaxation is much longer than those of cold collapse simulations. The effect of such sequence of merging on the LV criticality remains as a future work.

From astronomical points of view, observing the LV relation may give the hint for merging history of dark matter halos around galaxies. In fact, An and Evans proposed the general form of the phase-space distribution function following the LV relation and utilized them to evaluate the relation between the cusp slope of the dark matter halo in the galaxy and the anisotropy of γ-ray flux radiated from the galactic center [21]. Such observations may also reveal how well the LV relation is attained in dark matter halos around galaxies.

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APPENDIX: GLOBAL VIRIAL RELATION

In Eq. (6), R_a is a monotonic function of b for any fixed value of a. Hence if R_a is constant and independent of the position, b also becomes constant. This means that the local velocity dispersion σ^2 becomes proportional to the local potential everywhere. Assuming the spherical symmetry and describing Eq. (2) with the cumulative mass M_r as the coordinate in the radial direction, we obtain the relation

$$\sigma^2(M_r) = -b\Phi(M_r)/2. \tag{A1}$$

Integrating both sides of Eq. (A1) with M_r in the full of the bound region, we obtain the relationship between total kinetic energy K and total potential W as

$$2K = -bW, \tag{A2}$$

where K and W are the total kinetic energy and total potential defined as

$$K = \frac{1}{2}\int_0^{M_{\text{tot}}} \sigma^2(M_r)dM_r,$$

$$W = \frac{1}{2}\int_0^{M_{\text{tot}}} \Phi(M_r)dM_r, \tag{A3}$$

respectively. Hence the LV ratio b is identified with the global virial ratio V defined as

$$V = -\frac{2K}{W}, \tag{A4}$$

which means that b becomes equal to 1 for a globally virialized state.

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